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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) MM 5488-86-18	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Mechanics & Materials Ctr. Texas A&M University 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State and ZIP Code) College Station, Texas 77843	7b. ADDRESS (City, State and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION ONR 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER Contract N00014-86-K-0298
8c. ADDRESS (City, State and ZIP Code) Mechanics Division Office of Naval Research/Code 1513A:DLU 800 N. Qunicy Street Arlington, VA 22217-5000	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT NO.: 4324-520
11. TITLE (Include Security Classification) The Energy Release Rate for a Quasi-Static Mode I Crack in a Nonhomogeneous Linearly Viscoelastic Body	
12. PERSONAL AUTHOR(S) L. Schovance and J.R. Walton	
13a. TYPE OF REPORT Technical 13b. TIME COVERED FROM ______ TO ______	14. DATE OF REPORT (Yr., Mo., Day) September 1986 15. PAGE COUNT 21
16. SUPPLEMENTARY NOTATION	

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	viscoelasticity; dynamic fracture; dynamic energy release rate; Barenblatt zone; crack growth

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION	
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr. Richard L. Miller, Code 1132P	22b. TELEPHONE NUMBER (Include Area Code) (202) 696-4405	22c. OFFICE SYMBOL

DD FORM 1473, 83 APR

EDITION OF 1 JAN 73 IS OBSOLETE.

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Mechanics and Materials Center
TEXAS A&M UNIVERSITY
College Station, Texas

THE ENERGY RELEASE RATE FOR A QUASI-STATIC MODE I CRACK IN A NONHOMOGENEOUS LINEARLY VISCOELASTIC BODY

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OFFICE OF NAVAL RESEARCH
DEPARTMENT OF THE NAVY
CONTRACT N00014-86-K-0298
WORK UNIT 4324-520

MM-5488-86-18

SEPTEMBER 1986

The Energy Release Rate for a Quasi-Static Mode I Crack in a Nonhomogeneous Linearly Viscoelastic Body

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**Supported by the Office of Naval Research under Contract No. N00014-86-K-0298

Abstract

The Energy Release Rate (ERR) for the quasi-static problem of a semi-infinite mode I crack propagating through an inhomogeneous isotropic linearly viscoelastic body is examined. The shear modulus is assumed to have a power-law dependence on depth from the plane of the crack and a very general behavior in time. A Barenblatt type failure zone is introduced in order to cancel the singular stress and a formula for the ERR is derived which explicitly displays the combined influences of material viscoelasticity and inhomogeneity. The ERR is calculated for both power-law material and the standard linear solid and the qualitative features of the ERR are presented along with numerical illustrations.

1. Introduction.

In an earlier paper, [1], the quasi-static problem of a semi-infinite mode I crack propagating through an infinite inhomogeneous isotropic linearly viscoelastic body was investigated. The primary focus of that study was the description of the stress and displacement fields in a viscoelastic material characterized by a shear modulus exhibiting a very general behavior in time and a power-law spatial dependence. The principal objective of this paper is to calculate the Energy Release Rate (ERR) for the crack problem studied in [1] and to illustrate its dependence on the combined influences of material viscoelasticity and inhomogeneity.

In this work, the viscoelastic body in which the crack propagates is assumed to have a shear modulus of the form

$$\mu(t,y) = \mu_c g(t)(y/y_c)^\gamma \tag{1}$$

where y denotes distance as measured from the plane of the crack, y_c is a characteristic depth, $0 \le \gamma < 1$, and initially the only restrictions placed on g are that it be a positive, continuous, decreasing, and convex function of time. Subsequently, for illustrative purposes the specific forms of g to be considered will correspond to the standard linear solid and a power-law material.

The assumption of a simple power-law behavior in space for the shear modulus introduces certain unphysical features into the model. However, as pointed out in [1] and [2] and references cited therein, the consequences of the pure power-law model do not preclude its applicability to certain physical scenarios. Apart from the suitability of the shear modulus considered in this paper to any specific model, an important aspect of this work is the demonstration that for such a modulus, a tractable boundary value problem results, from which can be obtained meaningful average quantities associated with the energetics of the crack, specifically, the ERR. Furthermore, it shall be shown by a simple perturbation argument that for materials which experience a significant softening near the plane of the crack, a notion to be made precise later, the simpler assumption that the shear modulus vanishes in the plane of the crack may still provide a model for making acceptable engineering predictions.

The Stress Intensity Factor (SIF) and the ERR are familiar and important notions in the development of a fracture criterion. For a large class of crack problems in homogeneous, linear elastic materials, a simple relationship exists between the two quantities. However, in the case of crack propagation in an inhomogeneous viscoelastic material as considered here, the usual notion of the SIF is no longer valid since the order of the dominant singularity in the stress field near the crack tip is a function of the spatial inhomogeneity exponent. However, the notion of the ERR still provides a meaningful fracture criterion. Furthermore, by adopting the techniques and loading assumptions utilized in [3], the ERR is easily calculated and illustrates vividly the spatial and viscous effects.

The boundary value problem considered here is that of a mode I semi-infinite crack propagating with constant speed in a nonhomogeneous isotropic viscoelastic media in a state of plane strain. If the crack is assumed to propagate in a plane about which the spatial properties of the body are symmetric and along the x_1-axis with speed v, driven by loads $f(x_1-vt)$ which follow it, then the specific boundary value problem to be solved is

$$\sigma_{ij,j} = 0 \qquad -\infty < x_1 < \infty,\ x_2>0$$

$$\sigma_{ij} = 2\mu * d\varepsilon_{ij} + \delta_{ij}\left(\frac{2\nu}{1-2\nu}\right) \mu * d\varepsilon_{kk}$$

$$\sigma_{22}(x_1,0,t) = f(x_1-vt) \qquad x_1<vt$$

$$u_2(x_1,0,t) = 0 \qquad x_1>vt$$

$$\sigma_{12}(x_1,0,t) = 0 \qquad -\infty < x_1 < \infty.$$

Here σ_{ij}, ε_{ij}, and u_i are the viscoelastic stress, strain and displacement fields, ν is Poisson's ratio (assumed to be constant), μ is the shear modulus, δ_{ij} is the Kronecker delta, and $\mu * d\varepsilon$ denotes the Riemann-Stieltzes convolution,

$$\mu * d\varepsilon = \int_{-\infty}^{t} \mu(t-\tau)\, d\varepsilon(\tau).$$

After adopting the Galilean variable $x_1 = x-vt$ and the change of variables $y_1 = y$, $\sigma_{11} = \sigma_{xx}$, $\sigma_{12} = \sigma_{xy}$, $u_1 = u_x$, etc., it is useful to record for future reference the following results derived in [1]:

$$\hat{\sigma}_{yy}(p,0) = -C_0[ipv\hat{g}(-vp)]\, |p|^{1-\gamma}\, \hat{u}_y(p,0) \tag{2}$$

$$C_0 = \frac{\mu_c \Gamma(\gamma+2)\cos(\gamma\pi/2)}{q(1-\nu)\, I\, \sin(q\pi/2)\, y_c^{\gamma}}$$

$$q = \left[(1+\gamma)(1-\gamma\nu/(1-\nu))\right]^{1/2}$$

$$I = 2^{\gamma}(\gamma+2)\, B\left(\frac{\gamma+q+3}{2}, \frac{\gamma-q+3}{2}\right)$$

$$\sigma_{yy}^{+}(x) = \frac{\cos(\gamma\pi/2)}{\pi\ x^{(1-\gamma)/2}} \int_{-\infty}^{0} \sigma_{yy}^{-}(s)\, |s|^{(1-\gamma)/2} \frac{ds}{s-x},\ x>0. \tag{3}$$

From (3) it is easy to show that near the crack tip, as $x \to 0^+$, the singular term in the asymptotic expansion of the normal stress is given by

$$\sigma^+_{yy}(x) \sim \frac{K(\gamma)}{x^{(1-\gamma)/2}}$$

where

$$K(\gamma) = \frac{-\cos(\pi\gamma/2)}{\pi} \int_{-\infty}^{0} \frac{\sigma^-_{yy}(s)}{|s|^{(1+\gamma)/2}} ds. \qquad (4)$$

In the above equations the Fourier transform $\hat{f}$ of a function f is defined by

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, e^{ixp}\, dx,$$

with inverse

$$f^{\vee}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ixp}\, dx ,$$

and $f^+(x)(f^-(x))$ denote the restriction of $f(x)$ to $x>0(x<0)$.

In the next section of this paper a simple expression for the ERR will be derived for a general form of the time dependence in the shear modulus. The derivation of this formula will parallel, within a more general setting, the calculation of the ERR carried out in [3]. In the final section of the paper the ERR is calculated for some specific models and the qualitative features of the ERR are discussed. The paper concludes by illustrating, by means of a perturbation argument, the effect upon the ERR of modifying the spatial inhomogeneity in (1).

Section 2. The Calculation of the Energy Release Rate.

For fracture in linearly elastic material, the ERR, hereafter denoted by G, is usually defined as the difference between the total power input to the body by external forces, P, and the rate of change of total internal strain energy, $\dot{E}$, and kinetic energy, $\dot{K}$. G can then be calculated from knowledge of the singular asymptotic stress field at the crack tip. For a viscoelastic material the difference $P-\dot{E}-\dot{K}$ now involves a term representing the energy dissipated through viscous effects and the quantity G which depends upon the entire history of the singular asymptotic stress and strain fields at the crack tip. Because it is difficult to calculate G from the singular fields, a Barenblatt type failure zone is introduced behind the crack tip in order to cancel the singular asymptotic fields in front of the crack. More precisely, the boundary value problem presented above is modified by assuming that two loads are acting on the crack faces: the applied external tractions $\bar{\sigma}_{yy}(x,0) = f(x)$, and now denoted $\bar{\sigma}_e(x)$, and cohesive failure stresses $\bar{\sigma}_f(x)$ acting in a failure zone of length a_f immediately behind the crack tip. The only assumptions about $\bar{\sigma}_f(x)$ are that a_f is small relative to some length scale a_e associated with $\bar{\sigma}_e(x)$ and that if $K_e(\Upsilon)$ and $K_f(\Upsilon)$ are the coefficients given by (4) corresponding to $\bar{\sigma}_e$ and $\bar{\sigma}_f$ respectively, then $K_e(\Upsilon) + K_f(\Upsilon) = 0$. Hence the effect of the failure zone is to cancel the singular stresses ahead of the crack tip and thereby produce a cusp shaped crack profile behind the crack tip. What results is that for the steady-state problem considered here G is given by

$$G = -\int_{-a_f}^{0} \sigma_f^-(x)\, u_{y,x}^-(x,0)\, dx \tag{5}$$

where now $u_y(x,0)$ is the crack face displacement corresponding to the combined loading $\sigma_e^- + \sigma_f^-$.

A significant simplification in the calculation of (5) occurs for special cases of σ_e^- and σ_f^-. Specificially, the external load σ_e^- and failure zone stress, σ_f^-, will be assumed to have the forms

$$\sigma_e^-(x) = L_e e^{x/a_e},\ \sigma_f^-(x) = -L_f e^{x/a_f},\ -\infty < x < 0. \tag{6}$$

For $a_f/a_e \ll 1$, the essential features of the Barenblatt model are satisfied by the assumptions (6), namely, a set of cohesive stresses and associated length scale a_f and a length scale a_e associated with the applied load σ_e^- such that σ_f^- cancels the singular stresses produced by σ_e^-. When σ_f^- and σ_e^- are given by (6), (5) is replaced by

$$G = -\int_{-\infty}^{0} \sigma_f^-(x)\, u_{y,x}^-\,(x,0)\, dx. \tag{7}$$

A general method for calculating (7) begins with (2). If $T(p) = -C_0[ipv\hat{g}(-vp)]|p|^{1-\gamma}$, then (2) may be written as

$$\hat{\sigma}^+(p,0) = T(p)\,\hat{u}_y^-(p,0) - \hat{\sigma}^-(p,0). \tag{8}$$

With the introduction of the Barenblatt failure zone u_y^- represents the displacements corresponding to the combined external and cohesive loads, $\sigma^- = \sigma_{yy}^- = \sigma_e^- + \sigma_f^-$, and $\sigma^+ = \sigma_{yy}^+$ is the resultant stress ahead of the cracks. The solution of (8) is given in [1]. Specifically, if $F^{\pm}(z)$, functions analytic in the upper and lower complex half-plane, are defined by

$$F^{\pm}(z) = \frac{X^{\pm}(z)}{2\pi i} \int_{-\infty}^{\infty} \frac{-\hat{\sigma}^-(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z} \tag{9}$$

with

$$X^+(p) = T(p)\, X^-(p), \tag{10}$$

then

$$\hat{\sigma}^+(p) = \lim_{\mathrm{Im}(z)\to 0^+} F^+(z)$$

and

$$\hat{u}_y^-(p) = \lim_{\mathrm{Im}(z)\to 0^-} F^-(z).$$

Furthermore, $X^+(z)$ has a particularly simple analytic extension to the upper half-plane given by

$$X^+(x) = c\sqrt{g(0)}\; z^{(1-\gamma)/2}, \tag{11}$$

c a constant.

Applying the Parseval formula for the Fourier transform, (7) may be written as

$$G = -\int_{-\infty}^{\infty} \check{\sigma}_f^-(p)\ \hat{u}_{y,x}^-(p)\ dp.$$

From (6) it follows that

$$\hat{\sigma}_f^-(p) = \frac{-a_f L_f}{(1+ia_f p)}\ ,\ \check{\sigma}_f^-(p) = \frac{-a_f L_f}{2\pi(1-ia_f p)} \tag{13}$$

with similar expressions for $\hat{\sigma}_e^-$ and $\check{\sigma}_e^-$. Since $\hat{u}_y^-$ has an analytic extension to the lower half-plane, namely $F^-(z)$, and $-ip\hat{u}_y = \hat{u}_{y,x}$, it follows that $\hat{u}_{y,x}$ can also be extended analytically into the lower half-plane and from (13) it is clear that $\check{\sigma}_f^-$ has a meromorphic extension to the lower half-plane with a simple pole at $-i/a_f$. Hence the integral in (12) may be evaluated using residues from which it follows that

$$G = -\frac{L_f}{a_f} F^-(-i/a_f). \tag{14}$$

To evaluate $F^-(-i/a_f)$, begin by writing $F^+(z) = F_e^+(z) + F_f^+(z)$ where from (9)

$$F_e^+(z) = \frac{-X^+(z)}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{\sigma}_e^-(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z}\ . \tag{15}$$

The analogous formulas to (11) for $\hat{\sigma}_e^-$ and $\check{\sigma}_e^-$ show that $\hat{\sigma}_e^-$ has a simple pole at i/a_e and since $X^+(z)$ is analytic in the upper half-plane (15) may be evaluated using residues to give

$$F_e^+(z) = -X^+(z)\left(\frac{a_e L_e}{1+ia_e z}\right)\left(\frac{1}{X^+(z)} - \frac{1}{X^+(i/a_e)}\right)$$

and so

$$F_e^+(p) = -\hat{\sigma}_e^-(p)\left(1 - \frac{X^+(p)}{X^+(i/a_e)}\right).$$

Similarly it follows that

$$F_f^+(p) = -\hat{\sigma}_f^-(p)\left(1 - \frac{X^+(p)}{X^+(i/a_f)}\right).$$

Now one may write

$$\begin{aligned}\hat{\sigma}_e(p) &= \hat{\sigma}_e^+(p) + \hat{\sigma}_e^-(p)\\ &= \hat{F}_e^+(p) + \hat{\sigma}_e^-(p)\\ &= \hat{\sigma}_e^-(p)\,\frac{X^+(p)}{X^-(i/a_e)}\end{aligned}$$

and similarly

$$\hat{\sigma}_f(p) = \hat{\sigma}_f^-(p)\,\frac{X^+(p)}{X^+(i/a_f)}.$$

It now follows that

$$\begin{aligned}\hat{\sigma}(p) &= \hat{\sigma}_e(p) + \hat{\sigma}_f(p)\\ &= X^+(p)\left[\frac{a_e L_e}{X^+(i/a_e)(1+ia_e p)} - \frac{a_f L_f}{X^+(i/a_f)(1+ia_f p)}\right]. \qquad (16)\end{aligned}$$

From (4) and the requirement that $K_e(\gamma) + K_f(\gamma) = 0$ there results

$$a_e^{(1-\gamma)/2} L_e = a_f^{(1-\gamma)/2} L_f,$$

or equivalently,

$$\frac{L_e}{X^+(i/a_e)} = \frac{L_f}{X^+(i/a_f)}. \tag{17}$$

Thus (16) may be written as

$$\hat{\sigma}(p) = \frac{(a_e-a_f)\, L_e\, X^+(p)}{(1+ia_e p)(1+ia_f p)\, X^+(i/a_e)} . \tag{18}$$

Since $F^-(p) = \hat{u}_y^-(p) = \hat{\sigma}(p)/T(p)$, combining (14), (17), and (18) it follows that

$$G = \frac{(a_f-a_e)}{(a_f+a_e)} \frac{L_e^2}{2} \frac{X^-(-i/a_f)\, X^+(i/a_f)}{[X^+(i/a_e)]^2} . \tag{19}$$

Using (10) and (11) there finally results

$$G = \frac{(a_e-a_f)}{(a_e+a_f)} \frac{L_e^2}{2C_0}\, a_e^{(1-\gamma)} \frac{a_f}{v\hat{g}(iv/a_f)}. \tag{20}$$

It should be noted that in general, beginning with a relationship as in (8), G will be given by (19) where $X^{\pm}(z)$ are solutions to the equation (10). As shown in [3], the above derivation can be modified to produce an analog of (20) for more general loads of the form

$$\sigma_e^-(x) = L_e \int_0^\infty e^{tx/a_e}\, dh_e(t)$$

$$\sigma_f^-(x) = -L_f \int_0^\infty e^{tx/a_f}\, dh_f(t)$$

where $h_e(t)$ and $h_f(t)$ are arbitrary signed (not necessarily positive) measures restricted only to the extent that the required integrals converge. In particular, the above derivation for G corresponds to $dh_e(t) = dh_f(t) = \delta(t-1)$, the Dirac measure concentrated at $t=1$. However, for the more general loads, the analog of (20) presents more numerical complications in the calculation of G.

Section 3. Examples and Results.

Two special cases of $g(t)$ corresponding to the standard linear solid, $g(t) = 1 + \eta \exp(t/\tau)$, and a power-law material, $g(t) = 1 + (t/\tau)^{-n}$, where τ is a characteristic relaxation time will now be considered. If the Carson transform $\bar{g}(t)$ of the function $g(t)$ is defined by

$$\bar{g}(s) = g(0) + \int_0^\infty e^{-ts}\, dg(t),$$

and the following nondimensional parameters are introduced

$$\varepsilon = a_f/a_e,\ \delta = y_c/a_e,\ \alpha = \tau v/a_f,$$

then after recalling the definition of C_0 and noting that

$\bar{g}(ivp) = ivp\hat{g}(-vp)$ (20) becomes

$$G = \frac{(1-\varepsilon)}{(1+\varepsilon)} \frac{a_e L_e^2}{2\mu_c} \frac{(2\delta)^\gamma}{E(\gamma)} \frac{1}{\bar{g}(v/a_f)} \tag{21}$$

with

$$E(\gamma) = \frac{[\Gamma(\gamma+2)]^2 \cos(\gamma\pi(2)}{q(1-\nu)\sin(q\pi/2)\,\Gamma\left(\frac{\gamma+q+3}{2}\right)\Gamma\left(\frac{\gamma-q+3}{2}\right)} .$$

For the standard linear solid

$$\bar{g}(v/a_f) = \frac{1+\alpha(1+\eta)}{1+\alpha} \tag{22}$$

and for power-law material

$$\bar{g}(v/a_f) = 1+\alpha^n\,\Gamma(1-n). \tag{23}$$

The numerical calculation of the ERR from (21) is a simple matter, and certain qualitative features of G are immediate. The inhomogeneous elastic effects in (21) are contained in the term $(2\delta)^\gamma/E(\gamma)$ while $1/\bar{g}(v/a_f)$ represents the viscous effects. The purely elastic problem is recovered when the crack speed $v=0$ and the homogeneous problem corresponds to $\gamma=0$. Consequently, the qualitative and quantitative effects of adding material viscoelasticity and inhomogeneity to the basic elastic model are easily identified and studied.

From (21) it is easy to determine an "effective depth", y_1, such that the ERR predicted by the inhomogeneous material model equals the ERR predicted by a homogeneous model with shear modulus equal to the inhomogeneous modulus evaluated at $y = y_1$. For a homogeneous material with modulus given by the inhomogeneous model evaluated at an effective depth y_1, i.e., for which

$$\mu(t) = \mu_c (y_1/y_c)^{\gamma} g(t) = \mu_e g(t),$$

the ERR is given by (21) with $\gamma=0$ and μ_e substituted for μ_c. Hence

$$G = \left(\frac{1-\varepsilon}{1+\varepsilon}\right) \frac{a_e L_e^2}{2\mu_e (1-\nu)\, \bar{g}(v/a_f)} . \tag{24}$$

Equating G in (21) and (24) there results

$$y_1 = \frac{a_e}{2} \left(\frac{E(\gamma)}{1-\nu}\right)^{1/\gamma} .$$

To illustrate the result, $(y_1/a_e)^{\gamma}$ is graphed in Figure 1 when Poisson's ratio $\nu=.3$.

It is apparent that G is a monotonically increasing funcion of δ. This behavior is explained by noting that as δ increases, so does the distance from the plane of the crack at which the material is as rigid as the homogeneous material with modulus $\mu(t) = \mu_c g(t)$. Thus with increasing δ there is a larger region of 'soft' material about the plane of the crack. In previous studies of crack problems in nonhomogeneous elastic materials (c.f. [4] and the accompanying references) it was observed that with a decrease in rigidity, there is an attendant increase in crack instability. Since G has the interpretation of power available to the crack tip for propagating the crack, a monotone increasing dependence on δ is in qualitative agreement with these previous studies.

It follows immediately from (22) and (23) that the ERR is a monotone decreasing function of α and hence of crack speed. Regardless of which model is used, the qualitative differences in G due to viscous effects are similar. For this reason the normalized ERR, g, defined by

$$G = \left(\frac{1-\varepsilon}{1+\varepsilon}\right) \frac{a_e L_e^2}{2\mu_c} g$$

is displayed in Figure 2 only for the standard linear solid. Note that g is singular at $\gamma=1$ just as the derivation of (2) is valid only for $0 \leq \gamma < 1$. A final remark along these lines is that if a simple power-law behavior in time, i.e., $g(t) = (t/\tau)^{-n}$, is assumed, then $\bar{g}(v/a_f) = \alpha^n \Gamma(1-n)$. Thus the simpler power-law model provides good agreement with the more realistic power-law model in predicting the ERR, yet has the desirable feature of providing a model which leads to a boundary value problem in which field quantities are more easily calculated [1].

This paper concludes by addressing the effect of a small perturbation in the spatial inhomogeneity upon the ERR. This analysis is most easily carried out in terms of displacements. If $\langle u,v \rangle$ denotes the displacement field corresponding to the original boundary value problem discussed above, then by applying a Fourier transform in the variable x to the stress-strain laws, the equations of equilibrium and the boundary conditions, the boundary value problem corresponding to the modulus $\mu(t,y) = \mu_c (y/y_c)^\gamma\ g(t)$ may be written as

$$\begin{aligned}
&L[\hat{u},\hat{v}](p,y) + \frac{\gamma}{y} B[\hat{u},\hat{v}](p,y) = 0 && y>0,\ -\infty < p < \infty \\
&\lim_{y\to 0}(y/y_c)^\gamma\ B_1[\hat{u},\hat{v}](p,y) = 0 && -\infty < p < \infty \qquad (24)\\
&\lim_{y\to 0}(y/y_c)^\gamma\ B_2[\hat{u},\hat{v}](p,y) = f(p) && p<0 \\
&\hat{v}(p,0) = 0 && p>0
\end{aligned}$$

where

$$L[u,v] = \begin{bmatrix} \dfrac{\partial^2 \hat{u}}{\partial y^2} - \dfrac{ip}{1-2\nu}\dfrac{\partial \hat{v}}{\partial y} - \dfrac{2(1-\nu)p^2}{1-2\nu}\hat{u} \\[2ex] \dfrac{\partial^2 \hat{v}}{\partial y^2} - \dfrac{ip}{2(1-\nu)}\dfrac{\partial \hat{u}}{\partial y} - \dfrac{(1-2\nu)p^2}{2(1-\nu)}\hat{v} \end{bmatrix}$$

$$B_1[\hat{u},\hat{v}] = \frac{\partial \hat{u}}{\partial y}(p,y) - ip\ \hat{v}(p,y)$$

$$B_2[\hat{u},\hat{v}] = \frac{\partial \hat{v}}{\partial y}(p,y) - \frac{ip\nu}{(1-\nu)}\hat{u}(p,y)$$

$$f(p) = \frac{(1-2\nu)\ \hat{\sigma}_e^-(p)}{(1-\nu)\ 2\mu_c\ \bar{g}(i\nu p)}$$

and

$$B[u,v] = \begin{bmatrix} B_1(u,v) \\ B_2(u,v) \end{bmatrix} .$$

If the spatial inhomogeneity is modified as $\varepsilon + (y/y_c)^\gamma$, the above boundary value problem becomes

$$(y/y_c)^\gamma \left[L[\hat{u},\hat{v}] + (\gamma/y)\, B[\hat{u},\hat{v}]\right] + \varepsilon\, L[\hat{u},\hat{v}] = 0$$

$$\lim_{y\to 0}\left(\varepsilon + (y/y_c)^\gamma\right) B_1[\hat{u},\hat{v}] = 0 \qquad -\infty < p < \infty \qquad (25)$$

$$\lim_{y\to 0}\left(\varepsilon + (y/y_c)^\gamma\right) B_2[\hat{u},\hat{v}] = f(p) \qquad p<0$$

$$\hat{u}(p,0) = 0 \qquad p>0.$$

If the solution of (25) is expanded in the asymptotic series

$$\langle u,v\rangle = \sum_{n=0}^{\infty} \varepsilon^n \langle u_n, v_n\rangle,$$

then it is a straightforward matter to verify that substitution of this series into (25) shows that $\langle u_0, v_0\rangle$ solves the unperturbed problem (24) and hence corresponds to the solution obtained in [1]. Having found $\langle u_0, v_0\rangle$, one solves for $\langle u_n, v_n\rangle$ recursively. For example, $\langle u_1, v_1\rangle$ satisfies

$$(y/y_c]^\gamma \left[L[\hat{u}_1, \hat{v}_1] + \gamma/y\; B[\hat{u}_1, \hat{v}_1]\right] + L[\hat{u}_0, \hat{v}_0] = 0$$

$$\lim_{y\to 0}\left[B_1(\hat{u}_0, \hat{v}_0) + (y/y_c)^\gamma\, B_1[\hat{u}_1, \hat{v}_1]\right] = 0 \qquad -\infty < p < -\infty$$

$$\lim_{y\to 0}\left[B_2[\hat{u}_0, \hat{v}_0] + (y/y_c)^\gamma\, B_2[\hat{u}_1, \hat{v}_1]\right] = 0 \qquad p<0$$

$$\hat{u}_1(p,0) = 0 \qquad p>0.$$

The ERR corresponding to the perturbed problem (25) can now be expressed by the formal series

$$G = \int_{-\infty}^{\infty} \sigma_f{}^{\vee} \hat{v}_{,y} \, dp$$

$$= \int_{-\infty}^{\infty} \sigma_f{}^{\vee} \hat{v}_{0,y} \, dp + \varepsilon \int_{-\infty}^{\infty} \sigma_f{}^{\vee} \hat{v}_{1,y} \, dp + \varepsilon^2 \int_{-\infty}^{\infty} \sigma_f{}^{\vee} \hat{v}_{2,y} \, dp + \ldots$$

$$= G_0 + O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0$$

where G_0 corresponds to the ERR when the spatial inhomogeneity is given by $(y/y_c)^\gamma$. Thus when the spatial inhomogeneity is characterized by a modulus of the form $\varepsilon + (y/y_c)^\gamma$ and $\varepsilon \ll 1$, corresponding to a material that offers little resistance to shear near the crack plane, then the corresponding ERR agrees with the ERR predicted by the modulus (y/y_c^γ) to an order of ε.

References

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[2] J.R. Walton, The Sliding with Coulomb Friction of a Rigid Indentor Over a Power-Law Inhomogeneous Linearly Viscoelastic Half-Plane. J. App. Mech., Trans. ASME, 51, 289-293, (1984).

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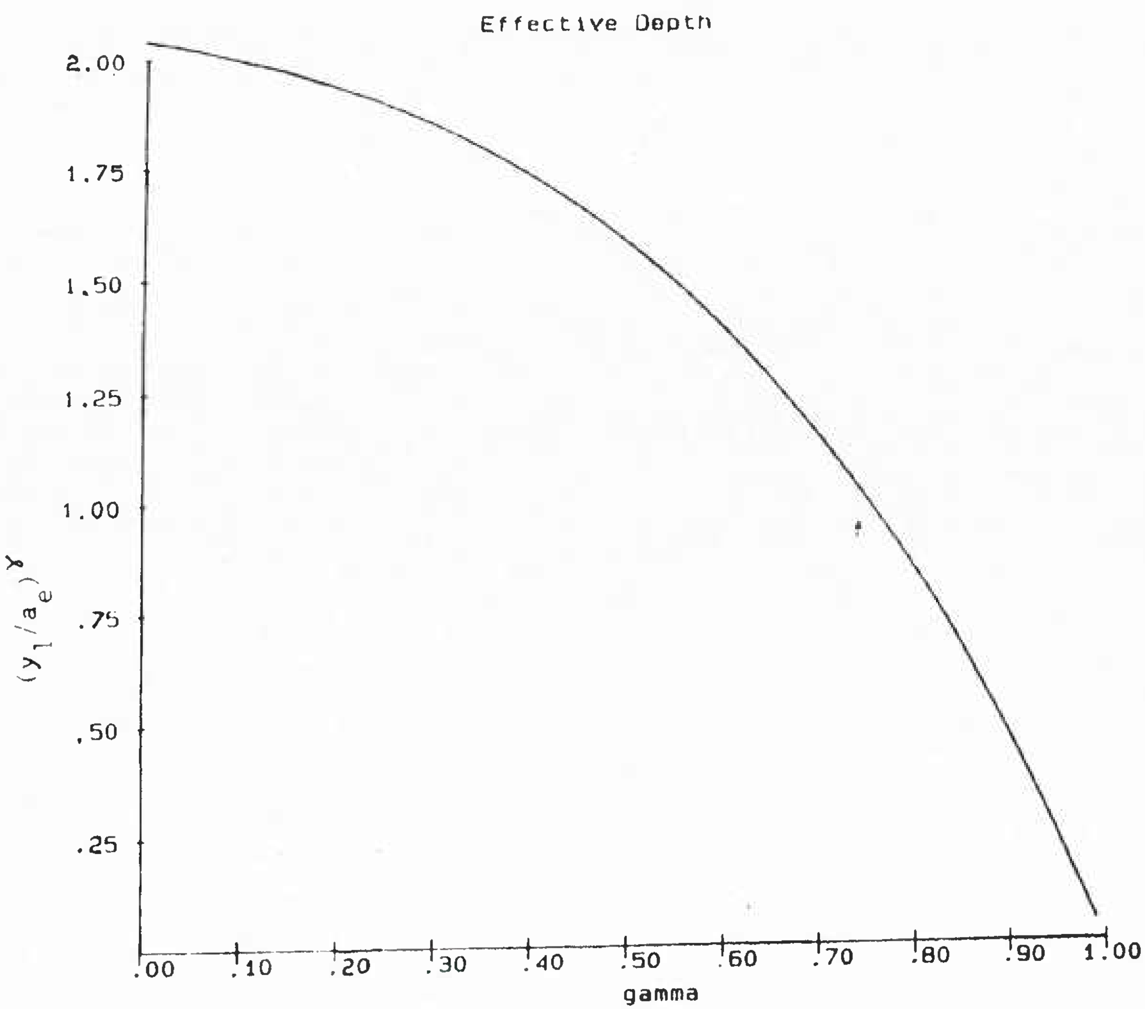
[4] L. Schovanec, A Griffith Crack Problem for an Inhomogeneous Elastic Material. Acta Mechanica, 58, 67-80 (1986).

Legends for Figures

Fig. 1. The normalized effective depth, $(y_1/a_e)^\gamma$.

Fig. 2. Normalized ERR for the standard linear solid with $\alpha=.1$, $\eta=10$, $\delta=.01$ (—), .1(— — —), 1(—·—).



Effective Depth
2.00
1.75
1.50
1.25
1.00
.75
.50
.25
$(y_1/a_e)^\gamma$
.00
.10
.20
.30
.40
.50
.60
.70
.80
.90
1.00
gamma

Figure 1

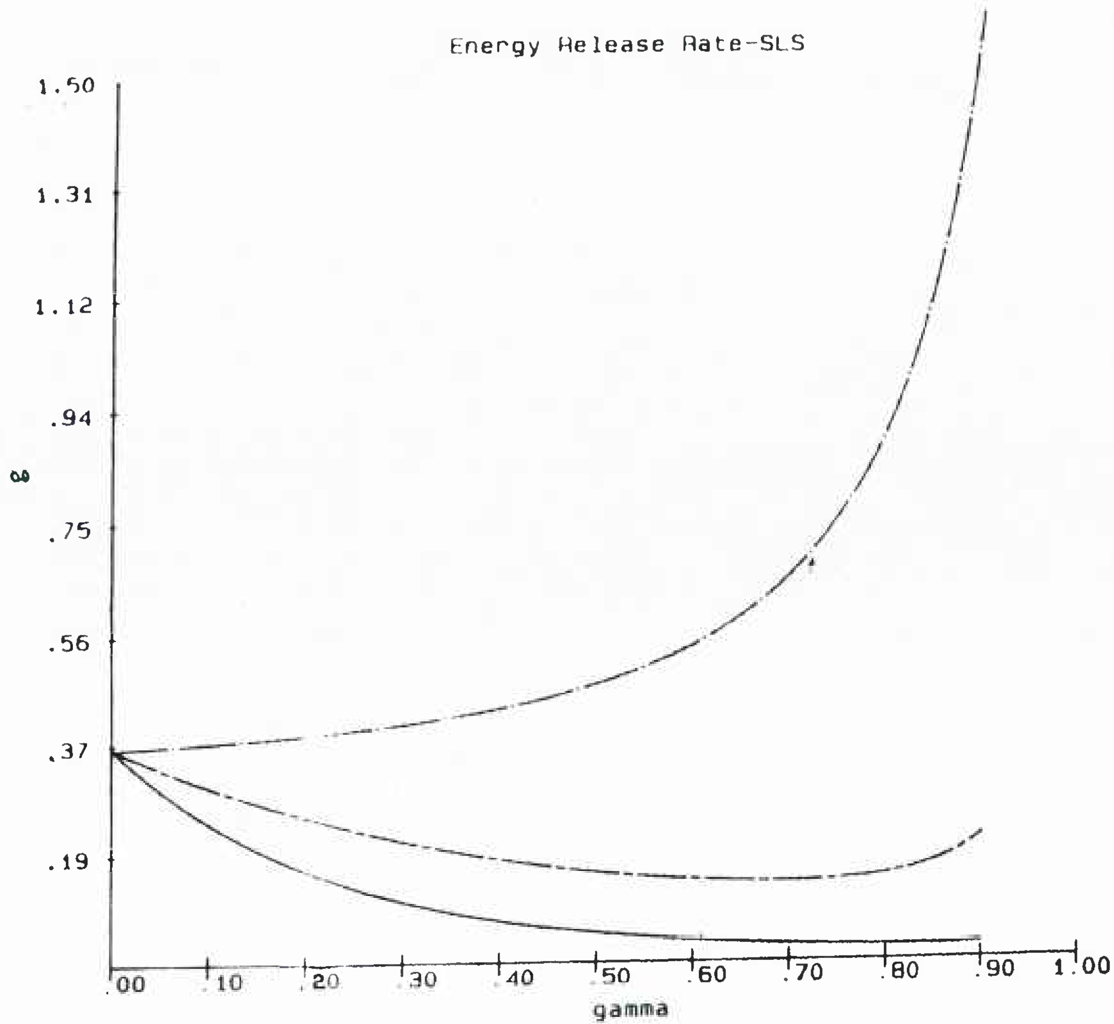

Energy Release Rate-SLS
1.50
1.31
1.12
.94
.75
.56
.37
.19
.00
.10
.20
.30
.40
.50
.60
.70
.80
.90
1.00
gamma

Figure 2